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NULL FIELDS REALIZATIONS OF W_3 FROM $W(sl(4), sl(3))$ AND $W(sl(3|1), sl(3))$ ALGEBRAS

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Abstract

We consider the nonlinear algebras $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ and find their realizations in terms of currents spanning conformal linearizing algebras. The specific structure of these algebras, allows us to construct realizations modulo null fields of the W_3 algebra that lies in the cosets $W(sl(4), sl(3))/u(1)$ and $W(sl(3|1), sl(3))/u(1)$. Such realizations exist for the following values of the W_3 algebra central charge: $c_W = -30, -40/7, -98/5, -2$. The first two values are listed for the first time, whereas for the remaining values we get the new realizations in terms of an arbitrary stress tensor and $u(1) \times sl(2)$ affine currents.

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1 Introduction

Nonlinear W algebras have been extensively considered from different points of view for the last years (see e.g. [1]). Despite many impressive results the question of constructing realizations is still open even for the simplest W_3 algebra. The well known tensoring procedure for constructing the new realizations starting from the known ones, which works perfectly in the case of linear algebras, cannot be applied to W algebras due to their intrinsic nonlinearity. That is why each new realization gives us a deeper understanding of the structure of nonlinear algebras.

One of the possible ways to construct the realizations of W algebras is the conformal linearization procedure [2–4]. The main idea of this approach is to embed nonlinear W algebra as a subalgebra in some linear conformal algebra W^{lin}. Once this is done, then each realization of the linear algebra W^{lin} gives rise to a realization of W. Recent results in this approach [4,5] show that fortunately a wide class of nonlinear algebras admits conformal linearization and therefore the realizations of these algebras can be constructed systematically.

However the problem of constructing the realizations of nonlinear algebras is far from its complete solution, due to the existence of the so called realizations modulo null fields [6–11]. The simplest example is given by the W_3 algebra, where we allow for the spin 4 null operator $U_4(z)$ in the operator product of two spin 3 generators $W(z)$:[1]

$$\begin{aligned} W(z_1)W(z_2) \; &= \; \frac{c_W}{3z_{12}^6} + \frac{2T}{z_{12}^4} + \frac{T'}{z_{12}^3} + \left[2U_4 + \frac{3}{10}T'' + \frac{32}{22+5c_W}\Lambda\right]\frac{1}{z_{12}^2} + \\ & \left[U_4' + \frac{1}{15}T''' + \frac{16}{22+5c_W}\Lambda'\right]\frac{1}{z_{12}} \; , \end{aligned} \tag{1.1}$$

with $\Lambda = (TT) - \frac{3}{10}T''$. For $U_4(z)$ to be a null operator we must require that there be no central term in the OPE of $U_4(z)$ with itself, so that $< U_4 U_4 >= 0$. The OPE (1.1) together with the standard OPEs with the Virasoro stress tensor $T(z)$ are not exactly the W_3 OPEs. Nevertheless $U_4(z)$, being a null operator, can only generate null fields in its OPEs. Therefore the set of null currents generated in a closed algebra is an ideal and can be consistently set to zero leaving us with the realization of W_3.

It is clear that the construction of the realizations modulo null fields is a rather complicated task and can be naturally divided in two parts.

Firstly, because the OPE (1.1) is a subset of the OPEs of some algebra $\mathcal{W}$, larger than W_3, we need to know this algebra, together with the condition for its contraction to W_3 (that is, the spectrum of the central charge that makes the equation $< U_4 U_4 >= 0$ satisfied). A first attempt to classify the possible $\mathcal{W}$ algebras which admit a contraction to W_N is made in [11], where a conjecture for the spectrum of central charges corresponding to the given contraction is proposed.

Secondly, we need to construct the realizations of the algebra $\mathcal{W}$ for the specific values of the central charge that allow the contraction of $\mathcal{W}$ to W_3. Up to now, such realizations of W_3 have been constructed for the following central charges: $c_W = -2, -114/7$ [8], $c_W = 4/5$ [7], $c_W = -10, -2, 4/5$ [10], $c_W = -98/5, -10, -2, 3/4$ [11], $c_W = -2, 4/5$ [9].

The aim of this Letter is to construct new realizations of the W_3 algebra modulo null fields

[1] The currents in the r.h.s. of the OPEs are evaluated at the point z_2 and $z_{12} = z_1 - z_2$ with normal ordering understood for the products of currents.

starting from the simplest nonlinear (super)algebras $W(sl(4),sl(3))$ and $W(sl(3|1),sl(3))$ which possess the following properties:

- the algebras include the OPE (1.1),
- the algebras can be linearized.

The linearizing algebras are useful for the construction of the realizations, while the presence of the OPE (1.1) guarantees the existence of the contraction to the W_3 algebra. New realizations exist for the following values of the W_3 central charge: $c_W = -40/7, -2, -30$ $(W(sl(4),sl(3)))^2$, $c_W = -2, -98/5$ $(W(sl(3|1),sl(3)))$. We would like to stress here that even for the previously known values of the central charge the constructed realizations of W_3 modulo null fields are new.

2 $W(sl(4),sl(3))$ and $W(sl(3|1),sl(3))$ algebras

In this Section we present the explicit structure of $W(sl(4),sl(3))$ and $W(sl(3|1),sl(3))$ algebras in the quantum case.

The $W(sl(4),sl(3))$ algebra can be constructed, as hinted by its very name, by considering the principal embedding of the $sl(2)$ algebra into the $sl(3)$ subalgebra of the $sl(4)$ [1,12]. The $W(sl(4),sl(3))$ algebra exists for a generic central charge c. Its currents obey the following OPEs:

$$
\begin{aligned}
T(z_1)T(z_2) &= \frac{2c}{z_{12}^4}+\frac{2T}{z_{12}^2}+\frac{T'}{z_{12}}, \quad T(z_1)U(z_2)=\frac{U}{z_{12}^2}+\frac{U'}{z_{12}}, \\
T(z_1)G(z_2) &= \frac{2G}{z_{12}^2}+\frac{G'}{z_{12}}, \quad T(z_1)\overline{G}(z_2)=\frac{2\overline{G}}{z_{12}^2}+\frac{\overline{G}'}{z_{12}}, \quad T(z_1)W(z_2)=\frac{3W}{z_{12}^2}+\frac{W'}{z_{12}}, \\
U(z_1)U(z_2) &= -\frac{c_1/2}{z_{12}^2}, \quad U(z_1)G(z_2)=-\frac{2G}{z_{12}}, \quad U(z_1)\overline{G}(z_2)=\frac{2\overline{G}}{z_{12}} \quad , \\
G(z_1)\overline{G}(z_2) &= \frac{c_2/3}{z_{12}^4}+\frac{a_1U}{z_{12}^3}+\left[\frac{1}{3}T+a_2(UU)+a_3U'\right]\frac{1}{z_{12}^2} \\
&\quad +\left[W+a_4(TU)+a_5(U'U)+\frac{1}{6}T'+a_6U''+a_7(UUU)\right]\frac{1}{z_{12}} \quad , \\
G(z_1)W(z_2) &= \frac{b_1G}{z_{12}^3}+\left[b_2(UG)+b_3G'\right]\frac{1}{z_{12}^2} \\
&\quad +\left[b_4(TG)+b_5(UUG)+b_6(U'G)+b_7(UG')+b_8G''\right]\frac{1}{z_{12}} \quad , \\
\overline{G}(z_1)W(z_2) &= -\frac{b_1\overline{G}}{z_{12}^3}+\left[b_2(U\overline{G})-b_3\overline{G}'\right]\frac{1}{z_{12}^2} \\
&\quad -\left[b_4(T\overline{G})+b_5(UU\overline{G})-b_6(U'\overline{G})-b_7(U\overline{G}')+b_8\overline{G}''\right]\frac{1}{z_{12}} \quad , \\
W(z_1)W(z_2) &= A\left\{\frac{c_W}{3z_{12}^6}+\frac{2\tilde{T}}{z_{12}^4}+\frac{\tilde{T}'}{z_{12}^3}+\left[2U_4+\frac{3}{10}\tilde{T}''+\frac{32}{22+5c_W}\Lambda\right]\frac{1}{z_{12}^2}\right.
\end{aligned}
$$

[2]Only two of these values $c_W = -40/7, -2$ coincide with the conjecture in [11].

$$+\left[U_4' + \frac{1}{15}\tilde{T}''' + \frac{16}{22+5c_W}\Lambda'\right]\frac{1}{z_{12}}\Bigg\}, \qquad (2.1)$$

where

$$\begin{aligned}
\tilde{T} &= T + \frac{1}{c_1}(UU), && (2.2)\\
\Lambda &= \tilde{T}\tilde{T} - \frac{3}{10}\tilde{T}'', && (2.3)\\
U_4 &= d_1(G\overline{G}) + d_2 W' + d_3(WU) + d_4(TT) + d_5(TUU) + d_6(T'U) + d_7(TU') \\
&\quad + d_8 T'' + d_9(UUUU) + d_{10}(U'UU) + d_{11}(U'U') + d_{12}(U''U) + d_{13}U'''. && (2.4)
\end{aligned}$$

and the values of all coefficients are given in the Table. We introduce the coefficient A in the r.h.s. of the $W(z_1)W(z_2)$ OPE in (2.1) for convenience. For sure, it can be set to one by rescaling the current $W(z)$, at the cost, however, of further complicating the coefficients in the Table.

Let us note that the spin 4 current $U_4(z)$ is defined to be primary with respect to the new stress tensor $\tilde{T}(z)$[3]

$$\tilde{T}(z_1)\tilde{T}(z_2) = \frac{c_W}{2z_{12}^4} + \frac{2\tilde{T}}{z_{12}^2} + \frac{\tilde{T}'}{z_{12}}, \quad \tilde{T}(z_1)U_4(z_2) = \frac{4U_4}{z_{12}^2} + \frac{U_4'}{z_{12}} \qquad (2.5)$$

and both $\tilde{T}$ and U_4 have regular OPEs with the $u(1)$ current $U(z)$

$$U(z_1)\tilde{T}(z_2) = U(z_1)U_4(z_2) = \text{regular}. \qquad (2.6)$$

Thus, the currents $\tilde{T}(z), W(z)$ and $U_4(z)$ belong to the coset $W(sl(4), sl(3))/u(1)$.

We would like also to stress that there is no possibility to redefine the currents of $W(sl(4), sl(3))$, in order to avoid the appearance of the $U_4(z)$ current in the r.h.s. of the OPE $W(z_1)W(z_2)$. Therefore, the $W(sl(4), sl(3))$ algebra does not contain the W_3 one as a subalgebra.

The $W(sl(3|1), sl(3))$ superalgebra contains currents with the same conformal spins as the $W(sl(4), sl(3))$ ones: the Virasoro stress tensor $T(z)$, a bosonic spin 1 current $U(z)$, a doublet of fermionic spin 2 currents $G(z)$ and $\overline{G}(z)$, and a bosonic spin 3 current $W(z)$. So the only differences in the contents of $W(sl(3|1), sl(3))$ and $W(sl(4), sl(3))$ algebras is the statistic of the spin 2 currents doublet. This is why the OPEs for the $W(sl(3|1), sl(3))$ algebra can be written in the form (2.1) with the same definitions for the composite currents (2.2)-(2.4) and the coefficients given in the Table.

[3]Due to the regular OPE of the spin 3 current $W(z)$ with $U(z)$, $W(z)$ is still a primary current, also with respect to $\tilde{T}$.

Coeff.	W(sl(4),sl(3))	W(sl(3|1),sl(3))	Coeff.	W(sl(4),sl(3))	W(sl(3|1),sl(3))
c	$\frac{c_1(4c_1+13)}{4(c_1-8)}$	$-\frac{c_1(2c_1+7)}{4(c_1+8)}$	b_1	$\frac{2(c_1+2)(5c_1-16)}{9c_1^2}$	$\frac{2(c_1-8)(c_1+2)(5c_1+16)}{9c_1^2(c_1+8)}$
c_2	$\frac{c_1(c_1+1)}{(c_1-8)}$	$-\frac{c_1(c_1+2)}{2(c_1+8)}$	b_2	$\frac{2(5c_1-16)}{3c_1^2}$	$\frac{2(c_1-8)(5c_1+16)}{3c_1^2(c_1+8)}$
c_W	$\frac{4(c_1+1)(c_1+2)}{c_1-8}$	$-\frac{2(c_1+2)^2}{c_1+8}$	b_3	$\frac{(c_1+8)(5c_1-16)}{18c_1^2}$	$\frac{(c_1-8)(5c_1+16)}{18c_1^2}$
a_1	$\frac{4(c_1+1)}{3(c_1-8)}$	$-\frac{2(c_1+2)}{3(c_1+8)}$	b_4	$\frac{4(c_1-8)}{3c_1(c_1+4)}$	$-\frac{8}{3c_1}$
a_2	$\frac{3}{c_1-8}$	$-\frac{1}{c_1+8}$	b_5	$\frac{4(5c_1-16)}{3c_1^2(c_1+4)}$	$\frac{8(c_1-16)}{3c_1^2(c_1+8)}$
a_3	$\frac{2(c_1+1)}{3(c_1-8)}$	$-\frac{c_1+2}{3(c_1+8)}$	b_6	$\frac{2}{c_1+4}$	$\frac{2}{c_1+8}$
a_4	$\frac{4}{3c_1}$	$\frac{4}{3c_1}$	b_7	$\frac{2(c_1-4)(c_1+16)}{3c_1^2(c_1+4)}$	$\frac{2(c_1-16)}{3c_1^2}$
a_5	$\frac{3}{c_1-8}$	$-\frac{1}{c_1+8}$	b_8	$\frac{c_1^3+4c_1^2+80c_1-256}{18c_1^2(c_1+4)}$	$\frac{(c_1-4)(c_1+16)}{18c_1^2}$
a_6	$\frac{2(c_1^2-2c_1+24)}{9c_1(c_1-8)}$	$-\frac{c_1^2+8c_1+48}{9c_1(c_1+8)}$	A	$-\frac{5c_1-16}{18c_1}$	$-\frac{(c_1-8)(5c_1+16)}{18c_1(c_1+8)}$
a_7	$\frac{4(11c_1-16)}{9c_1^2(c_1-8)}$	$-\frac{4(c_1-16)}{9c_1^2(c_1+8)}$			

Coeff.	W(sl(4),sl(3))	W(sl(3|1),sl(3))
d_1	$-\frac{72c_1}{(5c_1-16)(c_1+4)}$	$-\frac{144c_1}{(5c_1+16)(c_1-8)}$
d_2	$\frac{36c_1}{(5c_1-16)(c_1+4)}$	$\frac{72c_1}{(5c_1+16)(c_1-8)}$
d_3	$\frac{288}{(5c_1-16)(c_1+4)}$	$\frac{576}{(5c_1+16)(c_1-8)}$
d_4	$-\frac{12(c_1-8)(11c_1+20)}{(5c_1-16)(c_1+4)(10c_1^2+41c_1-68)}$	$\frac{24(c_1+8)(11c_1+20)}{(5c_1+16)(c_1-8)(5c_1^2+9c_1-68)}$
d_5	$-\frac{12(91c_1^2+260c_1-704)}{(5c_1-16)(c_1+4)(10c_1^2+41c_1-68)}$	$\frac{48(29c_1^2-36c_1-704)}{(5c_1+16)(c_1-8)(5c_1^2+9c_1-68)}$
$d_6=d_7$	$\frac{48}{(5c_1-16)(c_1+4)}$	$\frac{96}{(5c_1+16)(c_1-8)}$
d_8	$\frac{36(c_1^2-c_1+4)}{(5c_1-16)(10c_1^2+41c_1-68)}$	$\frac{36(c_1+4)(c_1^2+8)}{(5c_1+16)(c_1-8)(5c_1^2+9c_1-68)}$
d_9	$\frac{4(1153c_1^3+924c_1^2-19584c_1+25600)}{(c_1-8)c_1^2(5c_1-16)(c_1+4)(10c_1^2+41c_1-68)}$	$\frac{8(47c_1^3+1156c_1^2-1280c_1-25600)}{(5c_1+16)c_1^2(c_1-8)(c_1+8)(5c_1^2+9c_1-68)}$
d_{10}	$\frac{48(11c_1-16)}{(c_1-8)c_1(5c_1-16)(c_1+4)}$	$\frac{-96(c_1-16)}{(5c_1+16)c_1(c_1-8)(c_1+8)}$
d_{11}	$\frac{24(c_1-2)(23c_1^3+133c_1^2+140c_1+192)}{(c_1-8)c_1(5c_1-16)(c_1+4)(10c_1^2+41c_1-68)}$	$-\frac{24(7c_1^4+2c_1^3-220c_1^2-560c_1-768)}{(5c_1+16)c_1(c_1-8)(c_1+8)(5c_1^2+9c_1-68)}$
d_{12}	$\frac{4(178c_1^4+213c_1^3+1476c_1^2+17056c_1-32256)}{(c_1-8)c_1(5c_1-16)(c_1+4)(10c_1^2+41c_1-68)}$	$-\frac{8(31c_1^4+251c_1^3+1004c_1^2-4832c_1-32256)}{(5c_1+16)c_1(c_1-8)(c_1+8)(5c_1^2+9c_1-68)}$
d_{13}	$\frac{4(c_1^2-5c_1+48)}{(5c_1-16)(c_1-8)(c_1+4)}$	$\frac{4(c_1^2+14c_1+96)}{(5c_1+16)(c_1-8)(c_1+8)}$

Table.

3 Contractions of $W(sl(3|1), sl(3))$ and $W(sl(4), sl(3))$ to W_3 algebra

As stated in the Introduction, our purpose is building new realizations of W_3 modulo null fields and finding the corresponding values of the central charge. From the explicit OPEs of the $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ algebras (2.1) we can see that the currents $\tilde{T}(z)$ (2.2) and $W(z)$ form a W_3 algebra, modulo the spin 4 current $U_4(z)$ which is present in the r.h.s. of OPE $W(z_1)W(z_1)$. This spin 4 current U_4 in both cases is expressed in terms of the basic currents (2.4). Therefore we can require that the current $U_4(z)$ be a null operator, i.e. $< U_4 U_4 >= 0$. For this, all we need to ask is the vanishing of the central term in the $U_4(z_1)U_4(z_2)$ OPE. The corresponding equation is satisfied only for some special values of the central charge.

Next, we give the results for both $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$.

3.1 $W(sl(4), sl(3))$ case

Here we list for completeness the vacuum expectation values for the currents $\tilde{T}, W, U_4$ contained in the coset $W(sl(4), sl(3))/u(1)$

$$
\begin{aligned}
< \tilde{T}\tilde{T} > &= \frac{2(c_1+1)(c_1+2)}{(c_1-8)}, && (3.1)\\
< WW > &= -\frac{2(c_1+1)(c_1+2)(5c_1-16)}{27c_1(c_1-8)}, && (3.2)\\
< U_4U_4 > &= \frac{576(c_1-4)(c_1+1)(c_1+2)(c_1+6)(2c_1-1)}{(c_1-8)(c_1+4)(5c_1-16)(10c_1^2+41c_1-68)}. && (3.3)
\end{aligned}
$$

Thus for the values $c_1 = -6, \frac{1}{2}, 4$ which correspond to the following values of the W_3 central charge c_W:

$$
\begin{aligned}
c_1 = -6 \;&\Rightarrow\; c_W = -\frac{40}{7}, && (3.4)\\
c_1 = \frac{1}{2} \;&\Rightarrow\; c_W = -2, && (3.5)\\
c_1 = 4 \;&\Rightarrow\; c_W = -30, && (3.6)
\end{aligned}
$$

the spin 4 current U_4 becomes a null operator. All other poles and zeros of the vacuum expectation value $< U_4U_4 >$ provide us with further contractions of the algebra, where the spin 3 current $W(z)$ and even the stress tensor $\tilde{T}(z)$ become null operators.

Let us note that only the first two values of the W_3 central charge $c_W = -40/7, -2$ follow from the conjecture of [11]. Therefore the spectrum of central charges for the contraction of the $W(sl(4), sl(3))$ algebra to W_3 proposed in [11] is not exhaustive.

3.2 $W(sl(3|1), sl(3))$ case

The same calculations for the $W(sl(3|1), sl(3))$ superalgebra give the following results:

$$
\begin{aligned}
< \tilde{T}\tilde{T} > &= \frac{-(c_1+2)^2}{(c_1+8)}, && (3.7)\\
< WW > &= \frac{(c_1-8)(c_1+2)^2(5c_1+16)}{27c_1(c_1+8)^2}, && (3.8)\\
< U_4U_4 > &= -\frac{576(c_1-12)(c_1-1)(c_1+2)^2(c_1+4)^2}{(c_1-8)(c_1+8)^2(5c_1+16)(5c_1^2+9c_1-68)}. && (3.9)
\end{aligned}
$$

So, the current U_4 is a null operator for the following values:

$$
\begin{aligned}
c_1 = -4 \;&\Rightarrow\; c_W = -2, && (3.10)\\
c_1 = 1 \;&\Rightarrow\; c_W = -2, && (3.11)\\
c_1 = 12 \;&\Rightarrow\; c_W = -\frac{98}{5}. && (3.12)
\end{aligned}
$$

This concludes the determination of the central charges spectrum for both $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$, when these algebras are contracted to the W_3 one. In order to complete our task, we need to construct the realization of W_3 modulo null fields explicitly. In the next Section we will show that a straightforward way to construct such realizations comes from the conformal linearization procedure [2–4] applied to the algebras under consideration.

4 Null fields realizations of W_3 algebra

One of the most important questions when considering nonlinear W algebras is the construction of their realizations in terms of free fields or affine currents. The lack of a tensoring procedure for constructing new realizations starting from the known ones makes the task of finding new realizations of the nonlinear algebras rather difficult. Moreover, in many cases it is unclear which set of currents we need to use, in order to construct such realizations.

A solution to this problem was proposed in [2–4]. The idea of this approach is to embed the given nonlinear algebra into a *conformal linear* one which contains the former nonlinear algebra as a subalgebra. Of course, such linearizing algebra contains more currents, in comparison with the nonlinear one. However, after performing such a linearization, the question of constructing realizations becomes almost trivial, because any realization of the linear algebra gives rise to a realization of the corresponding nonlinear one.

4.1 Linearization of $W(sl(4), sl(3))$ algebra

Fortunately, for $W(sl(4), sl(3))$ such a linear algebra which contains it as a subalgebra is known [4]. It contains in the primary basis, besides the Virasoro stress tensor $\mathcal{T}(z)$, four currents with spin 1, i.e. $\mathcal{U}(z), \mathcal{J}_+(z), \mathcal{J}_3(z), \mathcal{J}_-(z)$ forming the $u(1) \times sl(2)$ affine algebra, and two additional currents $\mathcal{G}_1^+(z)$ and $\mathcal{G}_2^+(z)$ with the unusual spin $\frac{3K+4}{2K}$, where K is the level of the $u(1)$ affine algebra.[4] The complete list of OPEs for this linear algebra reads as follows:

$$
\begin{aligned}
\mathcal{T}(z_1)\mathcal{T}(z_2) &= \frac{(3-2K)(3K-4)}{2K z_{12}^4} + \frac{2\mathcal{T}}{z_{12}^2} + \frac{\mathcal{T}'}{z_{12}}, \\
\mathcal{T}(z_1)\mathcal{U}(z_2) &= \frac{\mathcal{U}}{z_{12}^2} + \frac{\mathcal{U}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{J}_+(z_2) = \frac{\mathcal{J}_+}{z_{12}^2} + \frac{\mathcal{J}_+{}'}{z_{12}}, \\
\mathcal{T}(z_1)\mathcal{J}_-(z_2) &= \frac{\mathcal{J}_-}{z_{12}^2} + \frac{\mathcal{J}_-{}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{J}_3(z_2) = \frac{\mathcal{J}_3}{z_{12}^2} + \frac{\mathcal{J}_3{}'}{z_{12}} \quad , \\
\mathcal{T}(z_1)\mathcal{G}_1^+(z_2) &= \frac{(3K+4)\mathcal{G}_1^+}{2K z_{12}^2} + \frac{\mathcal{G}_1^+{}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{G}_2^+(z_2) = \frac{(3K+4)\mathcal{G}_2^+}{2K z_{12}^2} + \frac{\mathcal{G}_2^+{}'}{z_{12}}, \\
\mathcal{U}(z_1)\mathcal{U}(z_2) &= \frac{K}{z_{12}^2}, \quad \mathcal{U}(z_1)\mathcal{G}_1^+(z_2) = \frac{\mathcal{G}_1^+}{z_{12}}, \quad \mathcal{U}(z_1)\mathcal{G}_2^+(z_2) = \frac{\mathcal{G}_2^+}{z_{12}}, \\
\mathcal{J}_+(z_1)\mathcal{J}_-(z_2) &= \frac{2-K}{z_{12}^2} - \frac{2\mathcal{J}_3}{z_{12}}, \quad \mathcal{J}_+(z_1)\mathcal{J}_3(z_2) = -\frac{\mathcal{J}_+}{z_{12}}, \quad \mathcal{J}_+(z_1)\mathcal{G}_1^+(z_2) = -\frac{\mathcal{G}_2^+}{z_{12}},
\end{aligned}
$$

[4]The currents of the $sl(2)$ affine algebra are related to those in the paper [4] as follows: $\mathcal{J}_3 = J_1^1, \mathcal{J}_+ = J_1^2, \mathcal{J}_- = -J_2^1$.

$$\begin{aligned}
\mathcal{J}_-(z_1)\mathcal{J}_3(z_2) &= \frac{\mathcal{J}_-}{z_{12}}, \quad \mathcal{J}_-(z_1)\mathcal{G}_2^+(z_2) = \frac{\mathcal{G}_1^+}{z_{12}}, \quad \mathcal{J}_3(z_1)\mathcal{J}_3(z_2) = \frac{K-2}{2z_{12}^2}, \\
\mathcal{J}_3(z_1)\mathcal{G}_1^+(z_2) &= -\frac{\mathcal{G}_1^+}{2z_{12}}, \quad \mathcal{J}_3(z_1)\mathcal{G}_2^+(z_2) = \frac{\mathcal{G}_2^+}{2z_{12}} \quad .
\end{aligned} \tag{4.1}$$

One may wonder how it could be possible to construct the currents of $W(sl(4), sl(3))$ from the currents of the linear algebra (4.1) which possess completely different spins. The answer becomes clear after defining the new stress tensor $T(z)$

$$T(z) = \mathcal{T} + \mathcal{J}_3' - \frac{K-2}{K}\mathcal{U}' \quad . \tag{4.2}$$

With respect to this stress tensor the currents of the linear algebra (4.1) have the following spins:

Currents	$\mathcal{U}$	$\mathcal{J}_+$	$\mathcal{J}_-$	$\mathcal{J}_3$	$\mathcal{G}_1^+$	$\mathcal{G}_2^+$
Spins	1	0	2	1	3	2

So, with respect to the stress tensor $T(z)$ (4.2) the currents of the linear algebra (4.1) possess the spins needed, in order to construct from them the currents forming the $W(sl(4), sl(3))$ algebra. Moreover, after imposing the following relation between the free parameter c_1 in $W(sl(4), sl(3))$ and the affine level K in the linear algebra:

$$c_1 = 8 - 6K \ , \tag{4.3}$$

the central charge for the stress tensor $T(z)$ (4.2) coincides with the central charge of the Virasoro subalgebra of (2.1). Therefore we can identify these two stress tensors (that is why we used the same letter in the definition (4.2)).

Now it is a matter of straightforward calculations to find the expressions for the remaining currents of the nonlinear $W(sl(4), sl(3))$ algebra in terms of the currents spanning the linear algebra (4.1)

$$\begin{aligned}
U &= \mathcal{U} + 2\mathcal{J}_3, \quad \mathcal{G} = \mathcal{J}_-, \\
\overline{G} &= \mathcal{G}_2^+ - \frac{(\mathcal{T}J_+)}{3(K-2)} - \frac{(\mathcal{J}_+\mathcal{J}_+\mathcal{J}_-)}{3K(K-2)} + \frac{2(\mathcal{J}_+\mathcal{J}_3\mathcal{J}_3)}{3K(K-2)} - \frac{(\mathcal{J}_+\mathcal{J}_3')}{3(K-2)} + \frac{2(\mathcal{U}\mathcal{J}_+\mathcal{J}_3)}{3K(K-2)} + \\
&\quad \frac{(\mathcal{U}\mathcal{U}\mathcal{J}_+)}{2K(K-2)} - \frac{2(\mathcal{U}\mathcal{J}_+')}{3K} - \frac{2(\mathcal{J}_+'\mathcal{J}_3)}{3K} + \frac{(1-K)(\mathcal{U}'\mathcal{J}_+)}{3K(K-2)} + \frac{(3-3K+K^2)\mathcal{J}_+''}{3K(K-2)}, \\
W &= \mathcal{G}_1^+ - \frac{(\mathcal{J}_+\mathcal{J}_-')}{3K} + \frac{4(5K-4)(\mathcal{J}_3\mathcal{J}_3\mathcal{J}_3)}{9(K-2)(3K-4)^2} + \frac{2(\mathcal{U}\mathcal{J}_+\mathcal{J}_-)}{3K(K-2)} - \frac{8(K^2-5K+4)(\mathcal{U}\mathcal{J}_3\mathcal{J}_3)}{3K(K-2)(3K-4)^2} + \\
&\quad \frac{(5K-4)(\mathcal{U}\mathcal{U}\mathcal{J}_3)}{3(K-2)(3K-4)^2} + \frac{(12-11K)(\mathcal{U}\mathcal{U}\mathcal{U})}{9K(3K-4)^2} + \frac{(4-K)(\mathcal{U}\mathcal{J}_3')}{3K(3K-4)} + \frac{2(1-K)(\mathcal{J}_+'\mathcal{J}_-)}{3K(K-2)} + \\
&\quad \frac{4(K^2-5K+4)(\mathcal{J}_3'\mathcal{J}_3)}{3K(K-2)(3K-4)} - \frac{K(\mathcal{U}'\mathcal{J}_3)}{3(K-2)(3K-4)} + \frac{(5K-4)(\mathcal{U}'\mathcal{U})}{6K(3K-4)} + + \frac{2(\mathcal{T}\mathcal{U})}{3(3K-4)} \\
&\quad \frac{(3K^3-22K^2+68K-48)\mathcal{J}_3''}{18K(K-2)(3K-4)} - \frac{(3K+2)\mathcal{U}''}{18(3K-4)} - \frac{2K(\mathcal{T}\mathcal{J}_3)}{3(K-2)(3K-4)} - \frac{\mathcal{T}'}{6}.
\end{aligned} \tag{4.4}$$

The expressions above, in spite of their rather complicated appearence, allow us to construct the realizations of the nonlinear $W(sl(4), sl(3))$ algebra starting from any given realization of the linear algebra (4.1). The simplest realization corresponds to the case when both the $\mathcal{G}_1^+$ and $\mathcal{G}_2^+$ currents are vanishing (i.e. they are null fields in the algebra (4.1)) and so we are left with the realizations of the nonlinear $W(sl(4), sl(3))$ algebra in terms of an arbitrary stress tensor $\mathcal{T}(z)$, and the $u(1) \times sl(2)$ affine currents $\mathcal{U}(z), \mathcal{J}_+(z), \mathcal{J}_-(z), \mathcal{J}_3(z)$.

Let us note that the three exceptional points $K = 0, 4/3, 2$, where the transformations (4.4) become singular, correspond to the central charges $c_1 = 8, 0, -4$ in the $W(sl(4), sl(3))$ algebra. At these points some of the coefficients in the Table are also singular and hence the currents that span the $W(sl(4), sl(3))$ algebra must be redefined, in order to avoid the singularities (see e.g. [8]). So, the appearance of singular terms in the transformations from the linearizing algebra (4.1) to the nonlinear $W(sl(4), sl(3))$ one is dictated by the structure relations for the $W(sl(4), sl(3))$ algebra we start with.

Thus, we succeed in the construction of the realizations (4.4) for $W(sl(4), sl(3))$. For the following values of the parameter: $K = 7/3, 5/4, 2/3$, which correspond to the central charge (3.4)-(3.6), the realization (4.4) is *the realization of* W_3 *modulo null fields.*

4.2 Linearization of $W(sl(3|1), sl(3))$ superalgebra

The conformal linearizing superalgebra for $W(sl(3|1), sl(3))$ has not been constructed so far. However the bosonic case we considered in the previous Section gives us some hints how such linear conformal superalgebra can be constructed. It is natural to assume that the linearizing superalgebra for $W(sl(3|1), sl(3))$ must contain the same spins as its bosonic counterpart (4.1) with some currents being now fermionic ones.

Without further justifications, we write down the linearizing superalgebra which contains the following currents: a Virasoro stress tensor $\mathcal{T}(z)$, four currents with spin 1, i.e. the bosonic currents $\mathcal{U}(z), \mathcal{J}_3(z)$ and the fermionic ones $\mathcal{J}_+(z), \mathcal{J}_-(z)$, as well as two additional fermionic currents $\mathcal{G}_1^+(z)$ and $\mathcal{G}_2^+(z)$ with spin $\frac{3K-32}{2(K-4)}$. The currents obey the OPEs

$$
\begin{aligned}
\mathcal{T}(z_1)\mathcal{T}(z_2) &= \frac{(K+11)(3K+8)}{10(K-4)z_{12}^4} + \frac{2\mathcal{T}}{z_{12}^2} + \frac{\mathcal{T}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{U}(z_2) = \frac{\mathcal{U}}{z_{12}^2} + \frac{\mathcal{U}'}{z_{12}}, \\
\mathcal{T}(z_1)\mathcal{J}_+(z_2) &= \frac{\mathcal{J}_+}{z_{12}^2} + \frac{\mathcal{J}_+{}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{J}_-(z_2) = \frac{\mathcal{J}_-}{z_{12}^2} + \frac{\mathcal{J}_-{}'}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{J}_3(z_2) = \frac{\mathcal{J}_3}{z_{12}^2} + \frac{\mathcal{J}_3{}'}{z_{12}}, \\
\mathcal{T}(z_1)\mathcal{G}_1^+(z_2) &= \frac{(3K-32)\mathcal{G}_1^+}{2(K-4)z_{12}^2} + \frac{\mathcal{G}_1^{+\prime}}{z_{12}}, \quad \mathcal{T}(z_1)\mathcal{G}_2^+(z_2) = \frac{(3K-32)\mathcal{G}_2^+}{2(K-4)z_{12}^2} + \frac{\mathcal{G}_2^{+\prime}}{z_{12}}, \\
\mathcal{U}(z_1)\mathcal{U}(z_2) &= \frac{K}{z_{12}^2}, \quad \mathcal{U}(z_1)\mathcal{G}_1^+(z_2) = \frac{\mathcal{G}_1^+}{z_{12}}, \quad \mathcal{U}(z_1)\mathcal{G}_2^+(z_2) = \frac{3\mathcal{G}_2^+}{z_{12}}, \\
\mathcal{U}(z_1)\mathcal{J}_+(z_2) &= \frac{2\mathcal{J}_+}{z_{12}}, \quad \mathcal{U}(z_1)\mathcal{J}_-(z_2) = -\frac{2\mathcal{J}_-}{z_{12}}, \quad \mathcal{U}(z_1)\mathcal{J}_3(z_2) = -\frac{K-4}{5z_{12}^2}, \\
\mathcal{J}_+(z_1)\mathcal{J}_-(z_2) &= -\frac{K-4}{10z_{12}^2} + \frac{\mathcal{J}_3}{z_{12}}, \quad \mathcal{J}_+(z_1)\mathcal{G}_1^+(z_2) = -\frac{\mathcal{G}_2^+}{z_{12}}, \\
\mathcal{J}_-(z_1)\mathcal{G}_2^+(z_2) &= \frac{\mathcal{G}_1^+}{z_{12}}, \quad \mathcal{J}_3(z_1)\mathcal{G}_1^+(z_2) = -\frac{\mathcal{G}_1^+}{z_{12}}, \quad \mathcal{J}_3(z_1)\mathcal{G}_2^+(z_2) = \frac{\mathcal{G}_2^+}{z_{12}} \quad , \qquad (4.5)
\end{aligned}
$$

where the free parameter c_1 in $W(sl(3|1), sl(3))$ and the affine level K in the linearizing superalgebra (4.5) are related as follows:

$$c_1 = \frac{8+3K}{5} \; . \tag{4.6}$$

Analogously to the bosonic case, we can express the $W(sl(3|1), sl(3))$ currents in terms of the currents of the linear superalgebra (4.5):

$$\begin{aligned}
T &= \mathcal{T} + \frac{2(K+1)}{K-4}\mathcal{J}_3' + \frac{1}{2}\mathcal{U}', \\
U &= \mathcal{U} + \mathcal{J}_3, \quad G = \mathcal{J}_-, \\
\overline{G} &= \mathcal{G}_2^+ + \frac{10(\mathcal{T}\mathcal{J}_+)}{3(K-4)} - \frac{75(\mathcal{J}_+\mathcal{J}_3\mathcal{J}_3)}{(K-4)^2} - \frac{10(\mathcal{J}_+\mathcal{J}_3')}{3(K-4)^2} + \frac{50(\mathcal{U}\mathcal{J}_+\mathcal{J}_3)}{(K-4)^2} + \frac{25(\mathcal{U}\mathcal{U}\mathcal{J}_+)}{3(K-4)^2} - \\
& \frac{10(K+6)(\mathcal{U}\mathcal{J}_+')}{3(K-4)^2} - \frac{20(K+11)(\mathcal{J}_+'\mathcal{J}_3)}{3(K-4)^2} - \frac{5(K+6))(\mathcal{U}'\mathcal{J}_+)}{(K-4)^2} + \frac{(K^2+7K+56)\mathcal{J}_+''}{3(K-4)^2}, \\
W &= \mathcal{G}_1^+ - \frac{100(\mathcal{J}_+\mathcal{J}_-\mathcal{J}_3)}{(K-4)^2} + \frac{10(\mathcal{J}_+\mathcal{J}_-')}{3(K-4)} + \frac{100(61K^2+327K+416)(\mathcal{J}_3\mathcal{J}_3\mathcal{J}_3)}{9(K-4)^2(3K+8)^2} + \\
& \frac{40(K+1)(\mathcal{T}\mathcal{J}_3)}{3(K-4)(3K+8)} + \frac{10(\mathcal{T}\mathcal{U})}{3(3K+8)} - \frac{100(\mathcal{U}\mathcal{J}_+\mathcal{J}_-)}{3(K-4)^2} + \frac{125(K+4)(11K+16)(\mathcal{U}\mathcal{J}_3\mathcal{J}_3)}{3(K-4)^2(3K+8)^2} + \\
& \frac{250K(K+6)(\mathcal{U}\mathcal{U}\mathcal{J}_3)}{3(K-4)^2(3K+8)^2} + \frac{25(K+16)(\mathcal{U}\mathcal{U}\mathcal{U})}{9(K-4)(3K+8)^2} - \frac{5(7K+32)(\mathcal{U}\mathcal{J}_3')}{6(K-4)(3K+8)} + \\
& \frac{20(K+6)(\mathcal{J}_+'\mathcal{J}_-)}{3(K-4)^2} - \frac{5(31K^2+32K-224)(\mathcal{J}_3'\mathcal{J}_3)}{6(K-4)^2(3K+8)} - \frac{5(7K^2+64K+32)(\mathcal{U}'\mathcal{J}_3)}{6(K-4)^2(3K+8)} - \\
& \frac{5(K+16)(\mathcal{U}'\mathcal{U})}{6(K-4)(3K+8)} - \frac{\mathcal{T}'}{6} + \frac{(K+1)(3K-52)\mathcal{J}_3''}{9(K-4)(3K+8)} + \frac{(3K^2-4K+368)\mathcal{U}''}{36(K-4)(3K+8)}.
\end{aligned} \tag{4.7}$$

The formulas (4.7) give us the desired realizations of the $W(sl(3|1), sl(3))$ superalgebra in terms of the currents of the linear superalgebra (4.5). For the values $K = -28/3, -1, 52/3$ this realization is *the realization of* W_3 *modulo null fields.*

5 Conclusion and outlook

In the present Letter we construct explicitly the nonlinear algebras $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ and find their realizations in terms of currents spanning the corresponding linearizing conformal algebras. The specific structure of these algebras allows us to construct modulo null fields realizations of the W_3 algebra that lies in the cosets $W(sl(4), sl(3))/u(1)$ and $W(sl(3|1), sl(3))/u(1)$. Such realizations exist for the following values of the W_3 algebra central charge: $c_W = -30, -40/7, -98/5, -2$. The first two values are listed for the first time, whereas for the last two values we get the new realizations in terms of an arbitrary stress tensor and $u(1) \times sl(2)$ affine currents.

Let us finish by presenting a conjecture about the spectrum of central charges for the minimal models of the considered $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ algebras. The idea can be described as follows: it is evident that for both linear algebras (4.1) and (4.5), after

putting the null currents $\mathcal{G}_1^+$ and $\mathcal{G}_2^+$ to zero, one can construct the stress tensor $T_d(z)$ which commutes with the remaining currents. The central charge c_d of this stress tensor $T_d(z)$ is still connected with the central charges of the nonlinear $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ algebras and the latter can be expressed in terms of c_d. This is the general situation for the conformal linearization procedure [2–4]. Moreover, in all known cases of linearizing algebras, the minimal models for the Virasoro algebra spanned by $T_d(z)$ reproduce the minimal models for the nonlinear algebras. One can assume that the same is true for the $W(sl(4), sl(3))$ and $W(sl(3|1), sl(3))$ algebras. If this conjecture is correct, then the central charge c_d for the Virasoro minimal models

$$c_d = 1 - 6\frac{(p-q)^2}{pq}$$

will induce the following values for the central charge of the minimal models:

$$c = \frac{(8p-15q)(4q-3p)}{pq} \quad \text{for } W(sl(4), sl(3)) \quad ,$$

and

$$c = \frac{(8p-3q)(2q-3p)}{pq} \quad \text{for } W(sl(3|1), sl(3)) \quad .$$

Of course, this conjecture must be checked by the standard methods.

An interesting extension of the results presented here comes from the consideration of the nonabelian case, which corresponds to the algebras $W(sl(N+3), sl(3))$ and $W(sl(3|N), sl(3))$. For these algebras, the same procedure for constructing the realizations of W_3 modulo null fields will give the series of the central charges (with a manifest dependence on N). This work is in progress.

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